IX. *On the laws of the polarization of light by refraction.* *By* DAVID BREWSTER, *LL.D. F.R.S. L. & E.*

Read Feb. 25, 1830.

IN the autumn of 1813 I announced to the Royal Society the discovery which I had then made of the polarization of light by refraction*; and in the November following I communicated an extensive series of experiments which established the general law of the phænomena. During the sixteen years which have since elapsed, the subject does not seem to have made any progress. From experiments indeed stated to have been performed at all angles of incidence with plates of glass, M. ARAGO announced that the quantity of light which the plate polarized by reflexion at any given angle was equal to the quantity polarized by transmission; but this result, founded upon incorrect observation, led to false views, and thus contributed to stop the progress of this branch of optics.

I had shown in 1813, from incontrovertible experiments, that the action of each refracting surface in polarizing light, produced a physical change on the refracted pencil, and brought it into a state approaching more and more to that of complete polarization. But this result, which will be presently demonstrated, was opposed as hypothetical by Dr. YOUNG and the French philosophers; and Mr. HERSCHEL has more recently given it as his decision, that of the two contending opinions, that which was first asserted by MALUS, and subsequently maintained by BIOT, ARAGO, and FRESNEL, is the most probable,—namely, that the unpolarized part of the pencil, in place of having suffered any physical change, retains the condition of common light.

I shall now proceed to apply to this subject the same principles which I

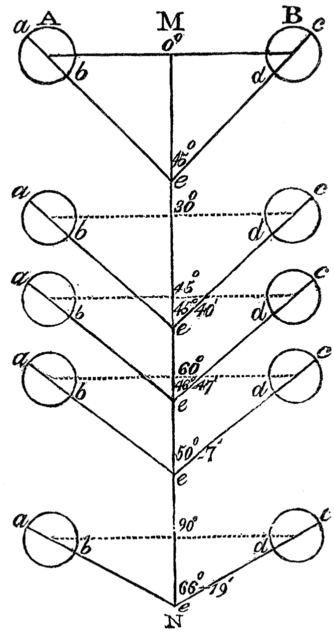
* In this discovery I was anticipated by MALUS.

have already applied to the polarization of light by reflexion, and to establish on the basis of actual experiment the true laws of the phenomena.

The first step in this inquiry is to ascertain the law according to which the polarizing force of the refracting surface changes the position of the planes of polarized light,—a subject which, in as far as I know, has not occupied the attention of any other person.

If we take a plate of glass deviating so slightly from parallelism as to throw off from the principal image the images formed by reflexion from its inner surfaces, we shall be able to see, even at great obliquities, the transmitted light free from all admixture of reflected light. Let this plate be placed upon a divided circle, so that we can observe through it two luminous discs of polarized light A, B (Fig. 1.) formed by double refraction, and having their planes of polarization inclined + 45° and — 45° to the plane of refraction. At an angle of incidence of 0°, when the light passes perpendicularly, the inclination of the planes of polarization will suffer no change; but at an incidence of 30° they will be turned round 40′; so that their inclination to M N or the angle *aec* will be 45° 40′. At 45° their inclination will be 46° 47′. At 60° it will be 50° 7′; and it will increase gradually to 90°, where it becomes 66° 19′. Hence the maximum change produced by a single plate of glass upon the planes of polarization is 66° 19′ — 45° = 21° 19′, an effect exactly equal to what is produced by reflexion at angles of 39 or 70°. It is remarkable, however, that this change is made in the opposite direction, the planes of polarization now approaching to coincidence in a plane at right angles to that of reflexion. This difference is exactly what might have been expected from the opposite character of the resulting polarization, the poles of the particles of light which were formerly repelled by the force of reflexion, being now attracted by the refracting force.

Fig. 1.

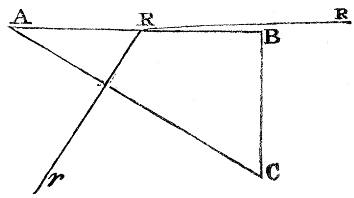


In this experiment the action of the two surfaces is developed in succession, so that we cannot deduce from the maximum rotation of 21° 19′, the real

action of the first, or of a single surface, which must be obviously more than half of the action of the two surfaces, because the planes of polarization have been widened before they undergo the action of the second surface.

In order to obtain the rotation due to a single surface, I took a prism of glass A B C (Fig. 2.) having such an angle B A C, that a ray RR, incident as obliquely as possible, should emerge in a direction R r perpendicular to the surface A C. I took care that this prism was well annealed, and I caused the refraction to be performed as near as possible to the vertex A, where the glass was thinnest and consequently most free from the influence of any polarizing structure. In this way I obtained the following measures.

Fig. 2.



GLASS.

Angles of Incidence.	Inclination of Planes ab, cd, (Fig. 1.) to the Plane of Reflexion.	Rotation.
87° 38′	54° 15′	9° 15′
54 50	47 25	2 25
32 20	45 22	0 22

I next made the following experiments with two kinds of glass,—the one a piece of parallel plate glass, and the other a piece of very thin crown. The latter had the advantage of separating the reflected from the transmitted light.

	PLATE GLASS.		CROWN GLASS.	
Incidence.	Inclination.	Rotation.	Inclination.	Rotation.
0°	45° 0′	0° 0′	45° 0′	0° 0′
40	47 28	2 28	47 18	2 18
55	49 35	4 35	49 19	4 19
67	52 53	7 53	52 16	7 16
80	58 53	13 53	58 42	13 42
86½	61 16	16 16	61 0	16 0

I was now desirous of ascertaining the influence of refractive power, although I had already determined in 1813, that a greater quantity of light was polarized, at the same angle of incidence, by plates of a high than by plates of a

low refractive power. I experienced great difficulty in this part of the inquiry, from the necessity of having plates without any crystalline structure. I tried gold leaf in a variety of ways; but I found it almost impossible to obtain correct results, on account of the light which was transmitted unchanged through its pores.

By stretching a film of soapy water across a rectangular frame of copper wire I obtained the following measure.

WATER.

Incidence.	Inclination.	Rotation.
85°	54° 17′	9° 17′

I next tried a thin plate of metalline glass of a very high refractive power.

METALLINE GLASS.

Incidence.	Inclination.	Rotation.
0°	45° 0′	0° 0′
20	45 42	0 42
30	46 50	1 50
40	48 0	3 0
55	51 12	6 12
80	62 32	17 32

From a comparison of these results it is manifest that the rotation increases with the refractive power.

In examining the effects produced at different angles of incidence, it becomes obvious that the rotation varies with the deviation of the refracted ray; that is, with $i - i'$ the difference of the angles of incidence and refraction. Hence from a consideration of the circumstances of the phænomena I have been led to express the inclination φ of the planes of polarization to the plane of refraction by the formula,

$$\text{Cot } \varphi = \cos (i - i'),$$

the rotation being $= \varphi - 45°$.

This formula obviously gives a minimum at 0°, and a maximum at 90°; and at intermediate points it represents the experiments so accurately, that when the rhomb of calcareous spar is set to the calculated angle of inclination, the extraordinary image is completely invisible,—a striking test of the correctness of the principle on which it is founded.

The above expression is of course suited only to the case where the inclina-

tion x of the planes of polarization ab, cd, (Fig. 1,) is 45°; but when this is not the case, the general expression is

$$\text{Cot } \varphi = \cot x \cos (i - i').$$

When the light passes through a second surface, as in a single plate of glass, the value of x for the second surface is evidently the value of φ after the 1st refraction, or in general, calling θ the inclination after any number n of refractions, and φ the inclination after one refraction,

$$\text{Cot } \theta = (\cot \varphi)^n$$

When θ is given by observation we have

$$\text{Cot } \varphi = \sqrt[n]{\cot \theta}.$$

The general formula for any inclination x and any number n of refractions is

$$\text{Cot } \theta = \left(\cot x \cos (i - i')\right)^n, \text{ and}$$

$$\text{Cot } \varphi = \sqrt[n]{\cot x \cos (i - i')}.$$

And when $x = 45$ and $\cot x = 1$ as in common light,

$$\text{Cot } \theta = \left(\cos (i - i')\right)^n.$$

$$\text{Cot } \varphi = \sqrt[n]{\cos (i - i')}.$$

As the term $\left(\cos (i - i')\right)^n$ can never become equal to 0, the planes of polarization can never be brought into a state of coincidence in a plane perpendicular to that of reflexion, either at the polarizing angle, or at any other angle.

In order to compare the formula with experiment, I took a plate of well annealed glass, which at all incidences separates the reflected from the transmitted rays, and in which m was nearly 1.510, and I obtained the following results.

Angles of Incidence.	Angles of Refraction.	Rotation observed.	Inclination observed.	Inclination calculated.	Difference.
0°	0° 0′	0° 0′	45° 0′	45° 0′	
10	6 36½	0 13	45 13	45 6	+0° 7′
20	13 5	0 27	45 27	45 25	+0 2
25	16 15	0 32	45 32	45 40	−0 8
30	19 20	0 40	45 40	46 0	−0 20
35	22 19	1 12	46 12	46 25	−0 13
40	25 10	1 30	46 30	46 56	−0 26
45	27 55	1 42	46 47	47 34	+0 47
50	30 29	2 48	47 42	48 24	−0 42
55	33 52	3 54	48 54	48 59	−0 5
60	35 0	5 7	50 7	50 36	−0 29
65	36 53	6 48	51 48	52 7	−0 19
70	38 29	8 7	53 7	53 59	−0 52
75	39 45	9 55	54 55	56 18	−1 23
80	40 42	12 10	57 10	59 5	−1 55
85	41 17	15 45	60 45	62 24	−1 39
86	41 21	16 39	61 39	63 9	−1 30
90	41 28			66 19	

The last column but one of the Table was calculated by the formula,

$$\text{Cot } \theta = \left(\cos\,(i - i')\right)^2$$

n being in this case 2. The conformity of the observed with the calculated results is sufficiently great, the average difference being only 41′. The errors however being almost all negative, I suspected that there was an error of adjustment in the apparatus; and upon repeating the experiment at 80°, the point of maximum error, I found that the inclination was fully 58° 40′, giving a difference only of 25′ in place of 1° 55′. I did not think it necessary to repeat all the observations; but I found, by placing the analysing rhomb at the calculated inclinations, that the extraordinary image invariably disappeared, the best of all proofs of the correctness of the formula.

In these experiments $x = 45°$ and cot $x = 1$; but in order to try the formula when x varied from 0° to 90°, I took the case where the angle of incidence was 80° and $\varphi = 58° 40'$ when $x = 45°$. The following were the results.

Values of x.	Inclination observed.	Inclination calculated.	Difference.
0°	0° 0′	0° 0′	0° 0′
$2\frac{1}{2}$	7 10	7 20	−0 10
5	9 40	8 19	+1 21
10	17 10	16 25	+0 45
15	24 42	24 6	+0 36
20	32 30	31 19	+1 11
25	39 15	37 54	+1 21
30	44 10	43 57	+0 13
35	49 38	49 28	+0 10
40	54 36	54 31	+0 5
45	58 40	59 5	−0 25
50	63 10	63 19	−0 9
55	66 58	67 15	−0 17
60	70 18	70 56	−0 38
65	74 8	74 24	−0 16
70	76 56	77 42	−0 46
75	79 20	80 53	−1 33
80	83 23	83 58	−0 35
85	86 23	86 0	+0 23
90	90 0	90 0	0 0

The last column but one was calculated by the formula $\cot \theta = \cot x \,.\, (\cot 58° \; 40')^2$. The differences on an average amount only to 36′.

In determining the quantity of polarized light in the refracted pencil, we must follow the method already explained for the reflected ray, *mutatis mutandis*. The principal section of the analysing rhomb being now supposed to be placed in a plane perpendicular to the plane of reflexion, the quantity of light Q′ polarized in that plane, will be

$$Q' = 1 - 2 \cos {}^2\varphi,$$

the quantity of transmitted light being unity. But

$$\text{Cot}\, \varphi = \cot x \cos (i - i'),$$

and as $\cot \varphi = \frac{\cos^2 \varphi}{\sin^2 \varphi}$ and $\sin^2 \varphi + \cos^2 \varphi = 1$, we have the quotient and the

sum of $\sin^2\phi$ and $\cos^2\phi$ to find them. Hence

$$\operatorname{Cos}^2\phi = \frac{(\cot x \cos(i - i'))^2}{1 + (\cot x \cos(i - i'))^2}$$

and by substituting this for $\cos^2\phi$ in the former equation, it becomes

$$Q' = 1 - 2\,\frac{(\cot x \cos(i - i'))^2}{1 + (\cot x \cos(i - i))^2}$$

Now since by FRESNEL's formula the quantity of reflected light is

$$R = \tfrac{1}{2}\left(\frac{\sin^2(i - i')}{\sin^2(i + i')} + \frac{\tan^2(i - i')}{\tan^2(i + i')}\right)$$

the quantity of transmitted light T will be

$$T = 1 - \tfrac{1}{2}\left(\frac{\sin^2(i - i')}{\sin^2(i + i')} + \frac{\tan^2(i - i')}{\tan^2(i + i')}\right)$$

Hence

$$Q' = \left(1 - \tfrac{1}{2}\left(\frac{\sin^2(i - i')}{\sin^2(i + i')} + \frac{\tan^2(i - i')}{\tan^2(i + i')}\right)\right)\left(1 - 2\,\frac{(\cos(i - i'))^2}{1 + (\cos(i - i'))^2}\right)$$

This formula is applicable to common light in which $\cot x = 1$ disappears from the equation; but on the same principles which we have explained in a preceding paper, it becomes for partially polarized rays and for polarized light,

$$Q' = \left(1 - \tfrac{1}{2}\left(\frac{\sin^2(i - i')}{\sin^2(i + i')}\cos^2 x + \frac{\tan^2(i - i')}{\tan^2(i + i')}\sin^2 x\right)\right)\left(1 - 2\,\frac{(\cot x \cos(i - i'))^2}{1 + (\cot x \cos(i - i'))^2}\right)$$

In all these cases the formula expresses the quantity of light really or apparently polarized in the plane of refraction.

As the planes of polarization of a pencil polarized $+ 45°$ and $- 45°$ cannot be brought into a state of coincidence by refraction, the quantity of light polarized by refraction can never be mathematically equal to the whole of the transmitted pencil, however numerous be the refractions which it undergoes; or, what is the same thing, refraction cannot produce rays truly polarized, that is, with their planes of polarization parallel.

The preceding analysis of the changes produced on common light, considered as represented by two oppositely polarized pencils, furnishes us with the same conclusions respecting the partial polarization of light by refraction, which we deduced in a preceding paper respecting the partial polarization of light by

reflexion. Each refracting surface produces a change in the position of the planes of polarization, and consequently a physical change upon the transmitted pencil by which it has approached to the state of complete polarization.

This position I shall illustrate by applying the formula to the experiments which I have published in the Philosophical Transactions for 1814.

According to the first of these experiments, the light of a wax candle at the distance of ten or twelve feet is wholly polarized by eight plates, or sixteen surfaces of parallel plate glass at an angle of 78° 52'. Now I have ascertained that a pencil of light of this intensity, will disappear from the extraordinary image, or appear to be completely polarized, provided its planes of polarization do not form an angle of less than $88\frac{3}{4}^\circ$ with the plane of refraction for a moderate number of plates, or $88\frac{1}{2}^\circ$ for a considerable number of plates, the difference arising from the great diminution of the light in passing through the substance of the glass. In the present case the formula gives

$$\text{Cot}\ \theta = \left(\cos\,(i - i')\right)^{16} \text{ and } \theta = 88^\circ\ 50'\,;$$

so that the light should appear to be completely polarized, as it was found to be.

At an angle of 61° 0' the pencil was polarized by 24 plates or 48 surfaces. Here

$$\text{Cot}\ \theta = \left(\cos\,(i - i')\right)^{48} = 89^\circ\ 36'.$$

At an angle of 43° 34' the light was polarized by 47 plates or 94 surfaces. Here

$$\text{Cot}\ \theta = \left(\cos\,(i - i')\right)^{94} \text{ and } \theta = 88^\circ\ 27'.$$

It is needless to carry this comparison any further; but it may be interesting to ascertain by the formula the smallest number of refractions which will produce complete polarization. In this case the angle of incidence must be 90°.

Hence $\phi = 56^\circ\ 29'$ and $\left(\cos\,(i - i')\right)^{9}$ gives 88° 36', and $\left(\cos\,(i - i)\right)^{10}$ 89° 4'; that is, the polarization will be nearly complete by the most oblique transmission through $4\frac{1}{2}$ plates or 9 surfaces, and will be perfectly complete through 5 plates or 10 surfaces.

Having thus obtained formulæ for the quantity of light polarized by refrac-

tion and reflexion, it becomes a point of great importance to compare the results which they furnish. Calling R the reflected light, these formulæ become

$$Q = R\left(1 - 2\,\frac{\left(\frac{\cos(i+i')}{\cos(i-i')}\right)^2}{1+\left(\frac{\cos(i+i')}{\cos(i-i')}\right)^2}\right) \text{ and}$$

$$Q' = 1 - R\left(1 - 2\,\frac{(\cos(i-i'))^2}{\phi + (\cos(i-i'))^2}\right).$$

But these two quantities are exactly equal, and hence we obtain the important general law, that,—At the first surface of all bodies, and at all angles of incidence, the quantity of light polarized by refraction is equal to the quantity polarized by reflection. I have said 'of all bodies', because the law is equally applicable to the surfaces of crystallized and metallic bodies, though the action of their first surface is masked or modified by other causes.

It is obvious from the formula that there must be some angle of incidence where $R = 1 - R$, that is, where the reflected is equal to the transmitted light. When this takes place, we have $\sin^2 \phi = \cos^2 \phi'$, that is,

The reflected is equal to the transmitted light, when the inclination of the planes of polarization of the reflected pencil to the plane of reflection, is the complement of the inclination of the planes of polarization of the refracted pencil to the same plane;—or if we refer the inclination of the planes to the two rectangular planes into which the planes of polarization are brought,—The reflected will be equal to the transmitted light when the inclination of the planes of polarization of the reflected pencil to the plane of reflection, is equal to the inclination of the plane of polarization of the refracted pencil to a plane perpendicular to the plane of reflection.

In order to show the connection between the phænomena of the reflected and those of the transmitted light, I have given the following Table, which shows the inclination of the planes of polarization of the reflected and the refracted pencil, and the quantities of light reflected, transmitted, and polarized, at all angles of incidence upon glass, *m* being equal to 1.525, and the incident light = 1000.

Angles of Incidence, i.	Angles of Refraction, i'.	Inclination of Plane of Polarization of the Reflected Light, φ'.	Inclination of Plane of Polarization of the Refracted Light, φ.	Quantity of Light Reflected, R.	Quantity of Light transmitted, 1 − R.	Quantity of Light Polarized, Q.
$0^\circ\ 0'$	$0^\circ\ 0'$	$45^\circ\ 0'$	$45^\circ\ 0'$	43.23	956.77	0.
2 0	1 $18\frac{2}{3}$	44 57	45 0.7	43.26	956.74	0.07
10 0	6 32	43 51	45 3	43.39	956.61	1.73
20 0	12 58	40 13	45 13	43.41	956.59	7.22
25 0	16 5	37 21	45 21	43.64	956.36	11.6
30 0	19 $8\frac{1}{2}$	33 40	45 31	44.78	955.22	17.24
35 0	22 6	29 8	45 44	46.33	953.67	24.4
40 0	24 56	23 41	46 0	49.10	950.90	32.2
45 0	27 $37\frac{1}{2}$	17 $22\frac{1}{2}$	46 20	53.66	946.33	44.0
50 0	30 9	10 18	46 45	61.36	938.64	57.4
56 45	33 15	0 0	47 29	79.5	920.5	79.5
60 0	34 36	5 $4\frac{1}{2}$	47 $54\frac{1}{2}$	93.31	906.69	91.6
65 0	36 28	12 45	48 42	124.86	875.14	112.7
70 0	38 2	18 32	49 28	162.67	837.33	129.8
75 0	39 18	26 52	50 55	257.56	742.44	152.3
78 0	39 54	30 44	51 48	329.95	670.05	157.6
78 7	39 55	30 53	51 50	333.20	666.80	157.65
79 0	40 4	31 59	52 7	359.27	640.73	157.6
80 40	40 13	33 13	52 $27\frac{1}{2}$	391.7	608.3	156.7
82 4	40 35	36 22	53 $26\frac{1}{3}$	499.44	500.56	145.4
84 0	40 42	38 2	53 57	560.32	439.68	134.93
85 0	40 47	39 12	54 22	616.28	383.72	123.7
85 $50\frac{2}{3}$	40 $50\frac{2}{3}$	40 12	54 44	666.44	333.56	111.11
86 0	40 51	40 $22\frac{7}{10}$	54 48	676.26	323.74	108.67
87 0	40 54	41 32	55 16	744.11	255.89	89.8
88 0	40 $57\frac{1}{2}$	41 23	55 43	819.9	180.1	65.9
89 0	40 58	43 51	56 14	904.81	95.19	36.3
90 0	40 58	45 0	56 29	1000.	0.	0.

It is obvious from a consideration of the principle of the formula for reflected light, that the quantity of polarized light is nothing at 0° because the force which polarizes it is there a minimum. At the maximum polarizing angle, Q is only 79° because the glass is incapable of reflecting more light at that angle, otherwise more would have been polarized. The value of Q then rises to its maximum at 78° 7′, and descends to its minimum at 90°; but the polarizing force has not increased from 56° 45′ to 78° 7′, as the value of φ' shows. It is only the quantity of reflected light that has increased, which occasions a greater quantity of light to disappear from the extraordinary image of the analysing rhomb.

The case however is different with the refracted light. The value of Q′ has one minimum at 0° and another at 90°, while its maximum is at 78° 7′,

while the force has its minimum at 0° and its maximum at 90°, where its effect is a minimum only because there is no light to polarize. At the incidence of 78° 7′, where the quantities Q , Q′ reach their maxima, the reflected light is exactly one half of the transmitted light; $\sin^2 \varphi' = \cos^2 \varphi$ and $\tan \varphi' = \cos \varphi$.

At 85° 50′ 40″, where the transmitted light is one half of the reflected light, the deviation $(i - i') = 45°$, and the quantity of polarized light is one third of the transmitted light, one sixth of the reflected light, and one ninth of the incident light. $\mathrm{Sin}^2 \varphi' : \cos^2 \varphi =$ reflected light : transmitted light, and $\cot \varphi' = \sin (i - i')$.

At 45° we have $(i + i') + (i - i') = 90°$ and $\varphi' = (i - i')$,

$$\mathrm{Tan}\ (i - i') = \frac{\cos (i + i')}{\cos (i - i')}, \text{ and } \tan (i - i')^2 = \frac{(\sin (i - i'))^2}{(\sin (i + i'))^2}$$

At 56° 45′, the polarizing angle, the formula for reflected light becomes $R = \frac{1}{2} \left(\sin^2 (i - i')\right)^2$; but at this angle we have $i' = 90° - i$. Hence we obtain the following simple expression in terms of the angle of incidence, for the quantity of light reflected by all bodies at the polarizing angle.

$$R = \tfrac{1}{2} (\cos 2i)^2.$$

I have already mentioned the experiment of M. Arago with plates of glass, in which he found that "at every possible inclination" the quantity of light polarized by transmission was equal to the quantity polarized by reflexion. This conclusion he extends to single surfaces; but it is remarkable that the law is true of single surfaces in which he did not ascertain it to be true, while it is incorrect with regard to plates in which he believes that he has ascertained it to be true. As the consideration of this point does not strictly belong to the present branch of the inquiry, I shall reserve it for a separate communication, "On the action of the second surfaces of transparent plates upon Light."

Allerly, December 29, 1829.